Math 246A Lecture 3 Notes

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October 1, 2018

1 Power Series and Analytic Functions

1.1 Power series

Lemma 1.1. *Suppose we have*

$$\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}|a_{n,k}| < \infty.$$

Then

$$\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}a_{n,k} = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}a_{n,k},$$

and both sums converge absolutely.

Theorem 1.1. *Given $z_0 \in \mathbb{C}$ and a sequence (a_n) in $\mathbb{C}$, let*

$$S(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n,$$

and let $1/R = \limsup |a_n|^{1/n} \in [0,\infty]$. Then

1. *if $r < R$, then $S(z)$ converges uniformly and absolutely in $\{z : |z-z_0| \le r\}$.*
2. *if $|z-z_0| > R$, then $S(z)$ does not converge.*

Proof. In the first case, let $r < s < R$. Then there exists N_0 such that $n \ge N_0 \implies |a_n|^{1/n} \le 1/s$. Then $|z-z_0| \le r$ and $n \ge N_0$ imply that $|a_n||z-z_0|^n \le (r/s)^n$. So

$$\sum_{n \ge N_0} |a_n||z-z_0|^n \le \sum_{n \ge N_0}\left(\frac{r}{s}\right)^n,$$

which converges because it is a geometric series with $|r/s| < 1$.

In the second case, $|z-z_0| > R \implies |z-z_0| > s > R$ for some s. Then $|a_n|^{1/n} \ge 1/s$ infinitely often. Then $|a_n||z-z_0|^n > 1$ infinitely often. So the series does not converge. $\square$

Theorem 1.2. *Assume* $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ *has radius of convergence* $R > 0$, *and let* $B = \{z : |z-z_0| < R\}$. *Set* $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ *for* $z \in B$.

1. *If* $z \in B$, *then there exists some sequence* (c_n) *(dependent on* z_0*) such that* $f(z) = \sum_{n=0}^{\infty} c_n(z-z_1)^n$ *when* $|z-z_1| < R - |z-z_0|$.

2. f *is complex differentiable in* B, *and* $f'(z) = \sum_{n=1}^{\infty} na_n(z-z_0)^{n-1}$.

Proof. Assume $|z-z_1| < r - |z-z_0| < R - |z-z_0|$. Then $|z-z_0| < r - |z-z_1| = s < R$, so $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ is bounded by

$$\sum_{n=0}^{\infty} |a_n| \left||z-z_0| + |z-z_1|\right|^n .$$

Since $|z-z_0| + |z-z_1| < r$, this converges. So

$$f(z) = \sum_{n=0}^{\infty} a_n \underbrace{\sum_{k=0}^{n} (z_1-z_0)^{n-k}(z-z_1)^k \binom{n}{k}}_{(z-z_0)^n},$$

which converges absolutely because

$$\sum_{n=0}^{\infty} |a_n| \sum_{k=0}^{n} |z_1-z_0|^{n-k}|z-z_1|^k \binom{n}{k} < \infty,$$

So by the lemma,

$$f(z) = \sum_{k=0}^{\infty} \underbrace{\left(\sum_{n=k}^{\infty} a_n \binom{n}{k} (z_1-z_0)^{n-k}\right)}_{c_k} (z-z_1)^k$$

converges. Note that

$$c_1 = \sum_{n=1}^{\infty} na_n(z_1-z_0)^{n-1}.$$

To prove part 2, without loss of generality, $z_1 = z_0$. Then

$$f(z) = f(z_0) + a_1(z-z_0) + \sum_{n=2}^{\infty} a_n(z-z_0)^n.$$

So $f(z) - (f(z_0) + a_1(z-z_0)) = o(|z-z_0|)$ because

$$\frac{|f(z) - (f(z_0) + a_1(z-z_0))|}{|z-z_0|} \le |z-z_0| \sum_{n=2}^{\infty} a_n |z-z_0|^{n-2}.$$

□

Corollary 1.1. *The functions* $f^{(k)}$ *are continuous and differentiable on* B *for all* $k \in \mathbb{N}$.

Proof. By induction. If $f^{(k-1)}$ satisfies the theorem, then $f^{(k)}$ is also a function satisfying the conditions of the above theorem. □

1.2 Analytic and holomorphic functions

Let $\Omega \subseteq \mathbb{C}$ be open and $f : \Omega \to \mathbb{C}$.

Definition 1.1. The function f is **analytic** on Ω ($f \in A(\Omega)$) if for all $z_0 \in \Omega$ there exists $B(z_0) = \{|z - z_0| < R_{z_0}\} \subseteq \Omega$ with $R_{z_0} > 0$ such that

$$f(z) = \sum_{n=0}^{\infty} a_n(z_0)(z - z_0)^n$$

for all $z \in B(z_0)$.

Recall that f is complex differentiable on Ω if $f'(z)$ exists for all $z \in \Omega$.

Definition 1.2. The function f is **holomorphic** on Ω if $f'(z)$ exists for all $z \in \Omega$ and $z \mapsto f'(z)$ is continuous.

We have shown that analytic implies holomorphic and, by definition, it is clear that holomorphic implies complex differentiable. We will later show that complex differentiable implies analytic.

Example 1.1. The following function has $R = 1$.

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}.$$

Example 1.2. The following function has $R = \infty$:

$$E(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

This function satisfies $E(z + w) = E(z) \cdot E(w)$ by the lemma. It also satisfies $E(0) = 1$, $E(-z) = 1/E(z)$, and $E'(z) = E(z)$. We also get that $E(t + i\theta) = E(t)E(i\theta)$ with $t, \theta \in \mathbb{R}$.

Lemma 1.2. *If $y : \mathbb{R} \to \mathbb{R}$ satisfies $y(0) = 1$ and $y' = y$, then $y = E(t)$.*

Proof. $E(t)$ satisfies the differential equation. Note that

$$\frac{d}{dt}(y(t)E(-t)) = \frac{y'(t)}{E(t)} - \frac{y(t)}{E(t)} = 0,$$

so $y = cE(t)$, and plugging in $y(0)E(-0) = 1$ gives us $y = E(t)$. $\square$

If $z = i\theta$ with $\theta \in \mathbb{R}$, then $E(\overline{z}) = \overline{E(z)}$. Define cosine and sine using $E(i\theta) = \cos(\theta) + i\sin(\theta)$. Using sine and cosine angle addition identities (which we get from $E(z + w) = E(z) \cdot E(w)$, we get $E(i\theta)E(-i\theta) = 1$. So $|E(i\theta)| = E(i\theta)\overline{E(i\theta)} = 1$.